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Unique Normal Form of Bogdanov–Takens Singularities

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In this note the method introduced by Kokubu, Oka, and Wang is improved and the unsolved problem in a paper of Baider and Sanders for the unique normal form of Bogdanov–Takens singularities is solved under a very general condition.

Key Words: grading function, normal form, Bogdanov–Takens singularity.

1. INTRODUCTION

Normal form theory has been studied for a long time (see, e.g., Arnold [Ar]) since it plays very important and elementary roles in the studies of

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bifurcation problems of vector fields. The basic idea of the classical normal form theory is to remove as many as possible terms by near identity transformations, while the near identity transformations are defined by one Lie bracket with the linear part of the given ordinary differential equation.

It has been noticed that the classical normal form theory may not give the simplest form and the normal form given by classical theory may not be unique in general, see for example, [KOW] and references therein. Here the nonuniqueness of normal forms means that the coefficients of normal forms may not be determined uniquely by the original equations even though the forms of the normal forms are fixed. Therefore the classical normal forms can not be used for formal classification of vector fields.

Ushiki [Us] used multiple Lie brackets involving nonlinear terms to get further reduction of the classical Normal forms. He gave the simplest normal forms up to some finite degrees for some vector fields. Wang [Wa] gave a method of computing the coefficients of normal forms from the original equations. He introduced parameters in the transformations so as to get simplest normal forms up to some finite degrees by setting parameters suitablely. Baider introduced the notion of "special form" [Ba], which is in fact the unique normal form in an abstract sense. Baider and Sanders [BS1] studied the unique normal forms for nilpotent Hamiltonian systems. They introduced the infinite order normal form in terms of new grading functions and proved the infinite order normal form is unique. They got unique normal forms for some nilpotent Hamiltonian vector fields. Then Baider and Sanders [BS2] introduced new grading function for planar vector fields to get further reduction of normal forms for Bogdanov–Takens singularites. The advantage of this method is the lowest homogeneous terms in the sense of the new grading contain not only linear terms but also some nonlinear terms in the sense of classical grading (i.e., the degree of a monomial is the sum of powers of all variables). They obtained unique normal forms for some cases of Bogdanov–Takens singularities. Results concerning uniqueness of normal forms for some other cases can be found in [BC2] and [SM]. Kokubu, Oka, and Wang ([KOW]) developed the method of new grading function and they defined nth order normal form in terms of n Lie brackets. They proved the infinite order normal form is unique and studied the condition under which the nth order normal form is in fact the infinite order one and hence is unique. They combined the techniques of new grading function and of the nth order normal form to solve a special case of the open problem in [BS2].

In the present paper, we apply the method introduced by [KOW] and improve some technique of computation, and solve the open problem in [BS2] under a very general condition.

2. PRELIMINARIES

In this section, we review some basic definitions and results about unique normal forms given by Kokubu, Oka, and Wang [KOW].

DEFINITION 2.1. Let

$$D_n = \left\{ \prod_{i=1}^{n} x_i^{l_i} \mathbf{e}_j \;\middle|\; l_i \in \mathbb{Z}^+,\, x_i \in \mathbb{R} \text{ (or } \mathbb{C}),\, i, j = 1, ..., n \right\},$$

where $\mathbf{e}_j$ is the jth standard unit vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$). Then the function $\delta\colon D_n \to \mathbb{Z}$ defined by

$$\delta\left(\prod_{i=1}^{n} x_i^{l_i} \mathbf{e}_j \right) = \sum_{i=1}^{n} a_i l_i - a_j, \tag{1}$$

where $a_i \in \mathbb{N}$, $i = 1, ..., n$, is called a linear grading function.

Remark 2.2. If we set all $a_i = 1$, then δ is the classical grading (shift by 1).

Let H_k be the linear space spanned by all monomials of degree k in the sense of grading function δ. Consider a formal vector field $V^{(0)}$ defined by the following formal series

$$V^{(0)} = V_\mu^{(0)} + V_{\mu+1}^{(0)} + \cdots + V_{\mu+k}^{(0)} + \cdots, \tag{2}$$

where $V_\mu^{(0)} \neq 0$ and $V_{\mu+k}^{(0)} \in H_{\mu+k}$, $k = 1, 2, ...$ We may assume that $V_\mu^{(0)}$ is already in some simple or satisfactory form (e.g., $V_\mu^{(0)}$ may have been changed to a simpler form by classical normal form theory).

For any $k \in \mathbb{N}$, define a linear operator

$$L_k^{(1)} : H_k \to H_{\mu+k}; \quad Y_k \mapsto [Y_k, V_\mu^{(0)}]. \tag{3}$$

Note that $L_k^{(1)}$ depends on $V_\mu^{(0)}$ and can be denoted by $L_k^{(1)} = L_k^{(1)}[V_\mu^{(0)}]$.

DEFINITION 2.3.

$$V = V_\mu + V_{\mu+1} + \cdots + V_{\mu+k} + \cdots$$

is called a *first order normal form*, if

$$V_{\mu+k} \in N_{\mu+k}^{(1)}, \qquad k = 1, 2, ...,$$

where $N^{(1)}_{\mu+k}$ is a complement subspace to $\operatorname{Im} L^{(1)}_k$ in $H_{\mu+k}$ and $L^{(1)}_k = L^{(1)}_k[V_\mu]$.

It is easy to see that there is a sequence of near identity formal transformations $\{y = \Phi_{Y_k}(x), k \in \mathbb{N}\}$ such that (2) is transformed into a first order normal form which is called the first order normal form of (2) and can be denoted by

$$V^{(1)} = V^{(1)}_\mu + V^{(1)}_{\mu+1} + \cdots + V^{(1)}_{\mu+k} + \cdots. \tag{4}$$

Note that $V^{(1)}_\mu = V^{(0)}_\mu$.

In order to make further reduction of a first order normal form, we define $L^{(2)}_k = L^{(2)}_k[V^{(1)}_\mu, V^{(1)}_{\mu+1}]$: $\operatorname{Ker} L^{(1)}_k \times H_{k+1} \to H_{\mu+k+1}$, $k = 1, 2, ...,$ by

$$L^{(2)}_k(Y_k, Y_{k+1}) = [Y_k, V^{(1)}_{\mu+1}] + [Y_{k+1}, V^{(1)}_\mu].$$

Then there is a sequence of near identity transformations $\{y = \Phi_{Y_k + Y_{k+1}}(x), k \in \mathbb{N}\}$ that send (4) to

$$V^{(2)} = V^{(2)}_\mu + V^{(2)}_{\mu+1} + V^{(2)}_{\mu+2} + \cdots + V^{(2)}_{\mu+k} + \cdots, \tag{5}$$

where $V^{(2)}_\mu = V^{(0)}_\mu$, $V^{(2)}_{\mu+1} = V^{(1)}_{\mu+1}$, $V^{(2)}_{\mu+k} \in N^{(2)}_{\mu+k}$ and where $N^{(2)}_{\mu+k}$ is a complement to $\operatorname{Im} L^{(2)}_{k-1}$ in $H_{\mu+k}$, $k = 2, 3, ...$ We call (5) the second order normal form. Then we can define a sequence of linear operators $L^{(m)}_k$, $m, k = 1, 2, 3, ...$ and the nth order normal form by induction.

Definition 2.4.

$$V = V_\mu + V_{\mu+1} + \cdots + V_{\mu+m} + \cdots$$

is called an *infinite order normal form*, if $V_{\mu+m} \in N^{(m)}_{\mu+m}$ for $\forall m \in \mathbb{N}$, where $N^{(m)}_{\mu+m}$ is a complementary subspace to $\operatorname{Im} L^{(m)}_1$ in $H_{\mu+m}$ and where $L^{(m)}_1 = L^{(m)}_1[V_\mu, V_{\mu+1}, ..., V_{\mu+m-1}]$ for $\forall m \in \mathbb{N}$.

Theorem 2.5. *The infinte order normal form of a given equation is unique.*

In some case, the infinite order normal form of the given equation may be the same as the Nth order one, namely, after one gets the Nth order normal form, then no term can be removed by using the further reduction technique introduced above. The following is the sufficient condition for the Nth order normal form to be the infinite order one.

THEOREM 2.6. *If there exists an $N \in \mathbb{N}$ such that*

$$\operatorname{Im} L_k^{(N+m)} = \operatorname{Im} L_{k+m}^{(N)} \tag{6}$$

for any $k, m \in \mathbb{N}$, then the Nth order normal form is an infinite order normal form.

Then the following lemma gives the sufficient condition for (6).

LEMMA 2.7. *If there exist $N, K \in \mathbb{N}$ such that*

$$\operatorname{Ker} L_k^{(N+1)} = \{0\} \times \operatorname{Ker} L_{k+1}^{(N)}, \qquad \forall k \geqslant K,$$

then $\operatorname{Im} L_k^{(N+m)} = \operatorname{Im} L_{k+m-1}^{(N+1)}$, $\forall k \geqslant K$ and $\forall m \in \mathbb{N}$

Remark 2.8. If the above lemma holds even for $N=0$, i.e., $\operatorname{Ker} L_k^{(1)} = \{0\}$, $\forall k \geqslant K$, then $\operatorname{Im} L_k^{(1+m)} = \operatorname{Im} L_{k+m}^{(1)}$, $\forall k \geqslant K$ and $\forall\ m \in \mathbb{N}$. But the condition $\operatorname{Ker} L_k^{(1)} = \{0\}$ can not hold for all $k \in \mathbb{N}$ in general since if $\mu \geqslant 1$ then $V_\mu^{(0)} \in \operatorname{Ker} L_\mu^{(1)}$ apparently.

3. THE BOGDANOV–TAKENS NORMAL FORM: THE CASE $\mu = 2\nu$

Baider and Sanders [BS2] gave unique normal forms for cases $\mu < 2\nu$ and $\mu > 2\nu$ of Bogdanov–Takens singularities. But they didn't solve the case $\mu = 2\nu$ completely. In [KOW] Kokubu, Oka, and Wang solved a special case $\nu = 1$, $\mu = 2$ under some restriction. In this section we consider general case $\mu = 2\nu$. By improving the method introduced by [KOW] we give the unique normal form under a very general condition, which includes the result given in [KOW].

We consider the following equation:

$$\begin{aligned} \dot{x} &= y + \text{h.o.t.}, \\ \dot{y} &= \alpha x^\nu y + \beta x^{2\nu+1} + \text{h.o.t.}, \end{aligned} \tag{7}$$

where $\nu \in \mathbb{N}$, $\alpha, \beta \neq 0$ and the h.o.t. denotes the higher order terms in the sense of the grading function δ defined below.

Define $\delta\colon D_2 \to \mathbb{Z}$ by

$$\delta\begin{pmatrix} x^m y^n \\ 0 \end{pmatrix} = m + n(\nu+1) - 1, \qquad \delta\begin{pmatrix} 0 \\ x^m y^n \end{pmatrix} = m + n(\nu+1) - \nu - 1.$$

Then δ is a linear grading function with

$$\delta\begin{pmatrix} y \\ 0 \end{pmatrix} = \delta\begin{pmatrix} 0 \\ x^{\nu}y \end{pmatrix} = \delta\begin{pmatrix} 0 \\ x^{2\nu+1} \end{pmatrix} = \nu.$$

Let

$$V_{\nu}^{(0)} = \begin{pmatrix} y \\ \alpha x^{\nu}y + \beta x^{2\nu+1} \end{pmatrix}.$$

Then the equation (7) can be written as

$$V^{(0)} = V_{\nu}^{(0)} + V_{\nu+1}^{(0)} + \cdots + V_{m}^{(0)} + \cdots \tag{8}$$

where $V_m^{(0)} \in H_m$, $m = \nu, \nu+1, \nu+2, \ldots$ and H_m is the linear space of homogeneous polynomials of degree m in the sense of δ.

LEMMA 3.1. *The following vectors form a basis of the space* H_m:
For $m = k(\nu+1)$, $k = 1, 2, \ldots,$

$$\begin{pmatrix} 0 \\ x^{(k+1)(\nu+1)} \end{pmatrix}, \begin{pmatrix} 0 \\ x^{k(\nu+1)}y \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ x^{\nu+1}y^{k} \end{pmatrix}, \begin{pmatrix} 0 \\ y^{k+1} \end{pmatrix},$$

$$\begin{pmatrix} x^{k(\nu+1)+1} \\ 0 \end{pmatrix}, \begin{pmatrix} x^{(k-1)(\nu+1)+1}y \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x^{(\nu+1)+1}y^{k-1} \\ 0 \end{pmatrix}, \begin{pmatrix} xy^{k} \\ 0 \end{pmatrix},$$

in this case $\dim H_m = 2k+3$;
For $m = k(\nu+1)+n$, $0 < n < \nu$, $k = 0, 1, 2, \ldots,$

$$\begin{pmatrix} 0 \\ x^{(k+1)(\nu+1)+n} \end{pmatrix}, \begin{pmatrix} 0 \\ x^{k(\nu+1)+n}y \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ x^{(\nu+1)+n}y^{k} \end{pmatrix}, \begin{pmatrix} 0 \\ x^{n}y^{k+1} \end{pmatrix},$$

$$\begin{pmatrix} x^{k(\nu+1)+1+n} \\ 0 \end{pmatrix}, \begin{pmatrix} x^{(k-1)(\nu+1)+1+n}y \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x^{(\nu+1)+1+n}y^{k-1} \\ 0 \end{pmatrix}, \begin{pmatrix} x^{1+n}y^{k} \\ 0 \end{pmatrix},$$

in this case, also $\dim H_m = 2k+3$;
For $m = k(\nu+1)+\nu$, $k = 0, 1, 2, \ldots,$

$$\begin{pmatrix} 0 \\ x^{(k+1)(\nu+1)+\nu} \end{pmatrix}, \begin{pmatrix} 0 \\ x^{k(\nu+1)+\nu}y \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ x^{(\nu+1)+\nu}y^{k} \end{pmatrix}, \begin{pmatrix} 0 \\ x^{\nu}y^{k+1} \end{pmatrix},$$

$$\begin{pmatrix} x^{(k+1)(\nu+1)} \\ 0 \end{pmatrix}, \begin{pmatrix} x^{k(\nu+1)}y \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x^{\nu+1}y^{k} \\ 0 \end{pmatrix}, \begin{pmatrix} y^{k+1} \\ 0 \end{pmatrix},$$

in this case $\dim H_m = 2k+4$.

From the above lemma, we can get the matrix representation for the adjoint operator $\mathrm{ad}(V_{\nu}^{(0)})$, i.e., $L_m^{(1)} = L_m^{(1)}[V_{\nu}^{(0)}]$. Note that $L_m^{(1)}$: $H_m \to H_{m+\nu}$;

$Y_m \mapsto [Y_m, V_v^{(0)}]$. Hence the size of the matrix representation L of $L_m^{(1)}$ is $(2k+4)\times(2k+3)$ for $m=k(v+1)$, $(2k+5)\times(2k+3)$ for $m=k(v+1)+n$, $0<n<v$ and $(2k+5)\times(2k+4)$ for $m=k(v+1)+v$. Let

$$L=\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix},$$

where the submatrix L_3 is such that $-L_3$ is the identity matrix of the size $l=k+2$ for any case where $m=k(v+1)+n$, $0\leqslant n\leqslant v$. The other three submatrices are given as follows:

For $m=k(v+1)$, $k=1, 2, ...,$

$$L_1=\begin{pmatrix} -\alpha & \beta & 0 & & & & \\ (k+1)(v+1) & 0 & 2\beta & & & \mathbf{0} & \\ & k(v+1) & \alpha & 3\beta & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & (k-1)\,\alpha & (k+1)\,\beta \\ \mathbf{0} & & & & v+1 & k\alpha \end{pmatrix};$$

$$L_2=\begin{pmatrix} -(2v+1)\,\beta & & & & & \\ -v\alpha & -(2v+1)\,\beta & & & & \mathbf{0} \\ & -v\alpha & -(2v+1)\,\beta & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & -v\alpha & -(2v+1)\,\beta \\ \mathbf{0} & & & & 0 & -v\alpha \end{pmatrix};$$

$$L_4=\begin{pmatrix} 0 & \beta & 0 & & & & \\ k(v+1)+1 & \alpha & 2\beta & & & \mathbf{0} \\ & (k-1)(v+1)+1 & 2\alpha & 3\beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & & (v+1)+1 & k\alpha \\ \mathbf{0} & & & & 0 & 1 \end{pmatrix}.$$

For $m=k(v+1)+n$, $0<n<v$, $k=0, 1, 2, ...,$

$$L_1=\begin{pmatrix} -\alpha & \beta & 0 & & & & \\ (k+1)(v+1)+n & 0 & 2\beta & & & \mathbf{0} \\ & k(v+1)+n & \alpha & 3\beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & & v+1+n & k\alpha \\ \mathbf{0} & & & & 0 & n \end{pmatrix};$$

$$L_2 = \begin{pmatrix} -(2v+1)\,\beta & & & & & \\ -v\alpha & -(2v+1)\,\beta & & & & \mathbf{0} \\ & -v\alpha & -(2v+1)\,\beta & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & -v\alpha & -(2v+1)\,\beta \\ \mathbf{0} & & & & 0 & -v\alpha \\ & & & & 0 & 0 \end{pmatrix};$$

$$L_4 = \begin{pmatrix} 0 & \beta & 0 & & & \\ k(v+1)+r & \alpha & 2\beta & & & \mathbf{0} \\ & (k-1)(v+1)+r & 2\alpha & 3\beta & & \\ & & \ddots & \ddots & & \ddots \\ & & & & (v+1)+r & k\alpha \\ \mathbf{0} & & & & 0 & r \end{pmatrix},$$

where $r = 1 + n$.

For $m = k(v+1) + v$, $k = 0, 1, 2, \ldots,$

$$L_1 = \begin{pmatrix} -\alpha & \beta & 0 & & & \\ (k+1)(v+1)+v & 0 & 2\beta & & & \mathbf{0} \\ & k(v+1)+v & \alpha & 3\beta & & \\ & & \ddots & \ddots & & \ddots \\ & & & & (v+1)+v & k\alpha \\ \mathbf{0} & & & & 0 & v \end{pmatrix};$$

$$L_2 = \begin{pmatrix} -(2v+1)\,\beta & & & & & \\ -v\alpha & -(2v+1)\,\beta & & & & \mathbf{0} \\ & -v\alpha & -(2v+1)\,\beta & & & \\ & & \ddots & \ddots & \ddots & \\ & & & & -v\alpha & -(2v+1)\,\beta \\ \mathbf{0} & & & & 0 & -v\alpha \end{pmatrix};$$

$$L_4 = \begin{pmatrix} 0 & \beta & 0 & & & \\ (k+1)(v+1) & \alpha & 2\beta & & & \mathbf{0} \\ & k(v+1) & 2\alpha & 3\beta & & \\ & & \ddots & \ddots & \ddots & \\ & & & & k\alpha & (k+1)\,\beta \\ \mathbf{0} & & & & v+1 & (k+1)\,\alpha \end{pmatrix}.$$

In order to simplify the expressions, we may assume that $\alpha = 1$ since we may make a suitable linear change of variables x, y, t in the equation (7) such that the coefficient of $x^v y$ is changed to 1 and the coefficient of x^{2v+1} is changed to β/α^2 accordingly.

LEMMA 3.2. *For $m = k(v+1)+n$, $0 \leqslant n \leqslant v$ the first $k+2+p$ rows of matrix L can be reduced to the form*

$$(0 \quad \tilde{M})$$

by suitable row transformations, where $p = 0$ if $n = 0$, $p = 1$ otherwise. Here the matrix

$$\tilde{M} = (M_{ij}) \qquad (-1 \leqslant i \leqslant k+p;\ 0 \leqslant j \leqslant k+q)$$

is given, where $q = 0$ if $n < v$, $q = 1$ if $n = v$, using $\alpha = 1$, as follows:

$$M_{i,\,i-1} = [(k+1-i)(v+1)+1+n][(k+1-i)(v+1)+n],$$
$$(i = 1, ..., k+p)$$
$$M_{i,\,i} = -v + i[(2k+1-2i)(v+1)+1+2n]$$
$$(i = 0, ..., k+p)$$
$$M_{i,\,i+1} = i(i+1) + \{(2i+3)[(k-i)(v+1)+n] - 3v + i\}\,\beta$$
$$(i = -1, 0, ..., k-1+p)$$
$$M_{i,\,i+2} = 2(i+1)(i+2)\,\beta \qquad (i = -1, ..., k-2+p)$$
$$M_{i,\,i+3} = (i+2)(i+3)\,\beta^2 \qquad (i = -1, ..., k-3+p)$$

and the other entries are all zero.

For convenience, we denote $M_{i,\,i-1} = a_i$, $M_{i,\,i} = b_i$, $M_{i,\,i+1} = c_i + d_i\beta$, $M_{i,\,i+2} = e_i\beta$, $M_{i,\,i+3} = f_i\beta^2$ for all cases.

LEMMA 3.3. *If β is not an algebraic number, then*

$$\operatorname{Ker} L_m^{(1)} = \{0\}, \qquad \forall m \in \mathbb{N},\ m \neq v.$$

To show the lemma, it is sufficient to show that

$$\det M \neq 0$$

where M is a submatrix of $\tilde{M}$ with the first row removed for the cases I and III or the first two rows removed for the case II. Since $\det M$ is a polynomial of β with interger coefficients, we only need to show that det M is not identically equal to zero, because β is not an algebraic number.

First we consider the case II. Since M is upper triangle, it is easy to show that

LEMMA 3.4. *In the case II, we have*

$$\det M = [k(v+1)+1+n]!^{v+1}\,[k(v+1)+n]!^{v+1},$$

where $m!^r = m(m-r)(m-2r)\cdots(m-lr)$ *and* l *is the maximal integer not bigger than* $(m-1)/r$.

Note that in case II, $k=0,\ 1,\ 2,\ \ldots,\ 0<n<v$. Therefore, det $M \neq 0$ for any β.

Then let us consider the case I.

LEMMA 3.5. *In the case I, we have*

$$\det M|_{\beta=0} = -v\cdot(k+1)!\cdot[k(v+1)-v]!^{v+1} \neq 0,$$

where $k=1, 2, \ldots$.

Proof. Let D_l be the following subdeterminant:

$$D_l = \det(M_{ij}|_{\beta=0})_{1\leqslant i,\, j\leqslant l}.$$

Then it is easy to see that

$$\det(M|_{\beta=0}) = (-v)\cdot D_k.$$

$D_1 = 2$ for $k=1$ obviously. By induction we can show that

$$D_l = \frac{(l+1)!\,[k(v+1)-v]!^{v+1}}{[(k-l)(v+1)-v)]!^{v+1}}, \qquad 1\leqslant l\leqslant k-1,$$

where $k\geqslant 2$. We denote by $D_0=1$ and $D_{-1}=0$. Then it follows that

$$\begin{aligned}\det M|_{\beta=0} &= -vD_k = -v\cdot(b_k\cdot D_{k-1} - c_{k-1}a_k\cdot D_{k-2})\\ &= -v\cdot(k+1)!\cdot[k(v+1)-v]!^{v+1}\neq 0,\end{aligned} \tag{9}$$

and hence the lemma is proved.

Next we consider the case III. We introduce the same subdeterminant

$$D_l = \det(M_{ij}|_{\beta=0})_{1\leqslant i,\, j\leqslant l}$$

for this case as well. Again, by induction, we can show

LEMMA 3.6. *For the case III, we have*

$$D_l = \frac{(l+1)!\,[k(\nu+1)]!^{\nu+1}}{[(k-l)(\nu+1)]!^{\nu+1}}, \qquad 1 \leqslant l \leqslant k.$$

where $k \geqslant 1$.

Remark 3.7. We note that $0!^{\nu+1}$ is defined to be equal to 1.

From the lemma, we have

$$\det(M|_{\beta=0}) = -\nu \cdot D_{k+1} = -\nu \cdot [b_{k+1} D_k - a_{k+1} c_k D_{k-1}] = 0.$$

Hence we cannot conclude $\det M \neq 0$ immediately. We therefore differentiate $\det M$ with respect to β and will show that

$$\frac{\partial}{\partial\beta} \det M|_{\beta=0} \neq 0.$$

Let $M^{(l)}$ be the matrix given by differentiating the lth row of the matrix M with respect to β at $\beta=0$. It thus takes the following form:

$$M^{(0)}|_{\beta=0} = \begin{pmatrix} 0 & 3k(\nu+1) & 4 & \cdots & \cdots & \cdots & \cdots & \cdots & & 0 \\ a_1 & b_1 & c_1 & & & & & & & \\ & \ddots & \ddots & \ddots & & & & \mathbf{0} & & \\ & & & a_l & b_l & c_l & & & & \\ & & & & & \ddots & \ddots & \ddots & & \\ & \mathbf{0} & & & & & a_k & b_k & c_k & \\ & & & & & & & a_{k+1} & b_{k+1} & \end{pmatrix}$$

$$M^{(l)}|_{\beta=0} = \begin{pmatrix} -\nu & 0 & & \cdots & \cdots & \cdots & \cdots & \cdots & & 0 \\ a_1 & b_1 & c_1 & & & & & & & \\ & \ddots & \ddots & \ddots & & & & \mathbf{0} & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & a_{l-1} & b_{l-1} & c_{l-1} & & & & \\ & & & & 0 & 0 & d_l & e_l & & \\ & & & & & a_{l+1} & b_{l+1} & c_{l+1} & & \\ & & & & & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & & & & & a_k & b_k & c_k \\ & & & & & & & & a_{k+1} & b_{k+1} \end{pmatrix}$$

where $0 < l \leqslant k$;

$$M^{(k+1)}|_{\beta=0} = \begin{pmatrix} -v & 0 & \cdots & \cdots & \cdots & 0 \\ a_1 & b_1 & c_1 & & & \\ & \ddots & \ddots & \ddots & & \mathbf{0} \\ & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & a_k & b_k & c_k \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

The determinants of these matrices are given as follows:

$$\det(M^{(0)}|_{\beta=0}) = (-a_1)[3k(v+1)\,\delta_k - 4a_2\delta_{k-1}],$$

$$\det(M^{(l)}|_{\beta=0}) = va_{l+1}D_{l-1}[d_l\delta_{k-l} - a_{l+2}e_l\delta_{k-l-1}], \qquad 1 \leqslant l \leqslant k,$$

$$\det(M^{(k+1)}|_{\beta=0}) = 0,$$

where

$$\delta_m = \det(M_{ij}|_{\beta=0})_{k+2-m \leqslant i,\, j \leqslant k+1}, \qquad 1 \leqslant m \leqslant k,$$

and

$$D_0 = \delta_0 = \delta_{-1} = 1, \qquad a_{k+2} = 0.$$

By induction, we can show:

Lemma 3.8.

$$\delta_m = \frac{k!\,[m(v+1)-1]!^{v+1}}{(k-m)!}, \qquad 1 \leqslant m \leqslant k.$$

From these formulas, we shall compute

$$\begin{aligned} &\frac{\partial}{\partial\beta} \det M|_{\beta=0} \\ &\quad = \sum_{l=0}^{k+1} M^{(l)} \\ &\quad = (-a_1)[3k(v+1)\,\delta_k - 4a_2\delta_{k-1}] \\ &\qquad + v \sum_{l=1}^{k-1} (a_{l+1}D_{l-1})[d_l\delta_{k-l} - a_{l+2}e_l\delta_{k-l-1}] + va_{k+1}d_kD_{k-1}. \end{aligned}$$

Let

$$\Delta_1 = (-a_1)[3k(v+1)\,\delta_k - 4a_2\delta_{k-1}],$$

$$\Delta_2 = v\sum_{l=1}^{k-1}(a_{l+1}D_{l-1})[d_l\delta_{k-l} - a_{l+2}e_l\delta_{k-l-1}],$$

$$\Delta_3 = va_{k+1}d_kD_{k-1}.$$

From Lemma 3.8, it is easy to get

$$\begin{aligned}\Delta_1 = &-(k+1)(v+1)[k(v+1)+v]\{3k(v+1)\,k!\,[k(v+1)-1]!^{v+1}\\ &-4k(v+1)[(k-1)(v+1)+v]\,k!\,[(k-1)(v+1)-1]!^{v+1}\}\\ =&\,k(v+1)^2\,(k+1)!\,[k(v+1)+v]!^{v+1},\end{aligned}$$

$$\begin{aligned}\Delta_3 &= v\cdot v(v+1)\cdot k(2v+1)\frac{k!\,[k(v+1)]!^{v+1}}{v+1}\\ &= kv^2(k!)^2\,(v+1)^k\,(2v+1).\end{aligned}$$

Now let us calculate Δ_2.

Lemma 3.9.

$$\begin{aligned}&\sum_{l=k-j}^{k-1}\frac{(v+1)^l}{(k-l)!}[(k-l)(v+1)+v]!^{v+1}\,[l(3v+2)-k(v+1)]\\ &\quad=\frac{(v+1)^{k-j}}{j!}(k-j)[(j+1)(v+1)+v]!^{v+1} - k(v+1)^k(2v+1)!^{v+1},\end{aligned}$$

where $1 \leqslant j \leqslant k-1$.

Proof. The lemma is proved by induction. It is easy to see that the lemma holds for $j=1$. Now we suppose that the lemma hold for j, where $1 \leqslant j < k-1$. Then

$$\begin{aligned}&\sum_{l=k-(j+1)}^{k-1}\frac{(v+1)^l}{(k-l)!}[(k-l)(v+1)+v]!^{v+1}\,[l(3v+2)-k(v+1)]\\ &\quad=\frac{(v+1)^{k-j-1}}{(j+1)!}[(j+1)(v+1)+v]!^{v+1}\,[(k-j-1)(3v+2)-k(v+1)]\\ &\qquad+\sum_{l=k-j}^{k-1}\frac{(v+1)^l}{(k-l)!}[(k-l)(v+1)+v]!^{v+1}\,[l(3v+2)-k(v+1)]\end{aligned}$$

$$= \frac{(v+1)^{k-j-1}}{(j+1)!}\,[(j+1)(v+1)+v]!^{v+1}\,[(k-j-1)(3v+2)-k(v+1)]$$

$$+ \frac{(v+1)^{k-j}}{j!}\,(k-j)[(j+1)(v+1)+v]!^{v+1} - k(v+1)^k\,(2v+1)!^{v+1},$$

$$= \frac{(v+1)^{k-j-1}}{(j+1)!}\,(k-j-1)[(j+2)(v+1)+v]!^{v+1}$$

$$- k(v+1)^k\,(2v+1)!^{v+1}.$$

Then it follows that

$$\Delta_2 = \sum_{l=1}^{k-1} v(k-l+1)(v+1)[(k-l)(v+1)+v]\,\frac{l!\,[k(v+1)]!^{v+1}}{[(k-l+1)(v+1)]!^{v+1}}$$

$$\times \Bigg\{[(k-l)(2l+3)(v+1)+l(2v+1)]\,\frac{k!\,[(k-l)(v+1)-1]!^{v+1}}{l!}$$

$$-(k-l)(v+1)[(k-l-1)(v+1)+v]$$

$$\times 2(l+1)(l+2)\,\frac{k!\,[(k-l-1)(v+1)-1]!^{v+1}}{(l+1)!}\Bigg\}$$

$$= \sum_{l=1}^{k-1} v(k!)^2\,\frac{(v+1)^l}{(k-l)!}\,[(k-l)(v+1)+v]!^{v+1}\,[l(3v+2)-k(v+1)]$$

$$= v(k!)^2\,\frac{v+1}{(k-1)!}\,\{[k(v+1)+v]!^{v+1} - k!\,(v+1)^{k-1}(2v+1)!^{v+1}\}.$$

Therefore

$$\frac{\partial}{\partial\beta}\det M|_{\beta=0}$$

$$= \Delta_1 + \Delta_2 + \Delta_3$$

$$= k(v+1)^2\,(k+1)!\,[k(v+1)+v]!^{v+1}$$

$$+ \frac{v(v+1)(k!)^2}{(k-1)!}\,\{[k(v+1)+v]!^{v+1} - k!\,(v+1)^{k-1}\,(2v+1)!^{v+1}\}$$

$$+ kv^2(k!)^2\,(v+1)^k\,(2v+1)$$

$$= k(v+1)\,k!\,[(k+1)(v+1)+v]!^{v+1}. \tag{10}$$

Hence $\det M \neq 0$ for case III if $k \neq 0$. And then we conclude that $\operatorname{Ker} L_m^{(1)} = \{0\},\ 1 \leqslant m \neq v$.

THEOREM 3.10. *If β/α^2 is not an algebraic number then the first order normal form of Eq. (7) is unique.*

Proof. The proof follows from Theorem 2.6. In fact the hypothesis of Theorem 2.6 holds for $N=1$ in this case. For any $k, m \in \mathbb{N}$,

$$\operatorname{Im} L_k^{(1+m)} = \{L_k^{(1+m)}(Y_k, Y_{k+1}, ..., Y_{k+m}) \mid (Y_k, Y_{k+1}, ..., Y_{k+m-1}) \in \operatorname{Ker} L_k^{(m)},\ Y_{k+m} \in H_{k+m}\} \tag{11}$$

Note that $(Y_k, Y_{k+1}, ..., Y_{k+m-1}) \in \operatorname{Ker} L_k^{(m)}$ if and only if

$$\begin{aligned} [Y_k, V_v] &= 0; \\ [Y_{k+1}, V_v] + [Y_k, V_{v+1}] &= 0; \\ &\vdots \\ [Y_{k+m-1}, V_v] + \cdots + [Y_k, V_{v+m-1}] &= 0. \end{aligned}$$

If $k > v$, then from above equations we have $Y_k = Y_{k+1} = \cdots = Y_{k+m-1} = 0$. And hence $\operatorname{Im} L_k^{(1+m)} = \{[Y_{k+m}, V_v] \mid Y_{k+m} \in H_{k+m}\} = \operatorname{Im} L_{k+m}^{(1)}$.
Since $\operatorname{Ker} L_v^{(1)} = \{eV_v \mid e \in \mathbb{R}\}$, it is easy to see that

$$\operatorname{Ker} L_v^{(m)} = \{(eV_v, eV_{v+1}, ..., eV_{v+m-1}) \mid e \in \mathbb{R}\}.$$

Note that

$$\begin{aligned} &[Y_{v+m}, V_v] + [eV_{v+m-1}, V_{v+1}] + [eV_{v+m-2}, V_{v+2}] + \cdots \\ &\quad + [eV_{v+2}, V_{v+m-2}] + [eV_{v+1}, V_{v+m-1}] + [eV_v, V_{v+m}] \\ &= [Y_{v+m} - eV_{v+m}, V_v]. \end{aligned}$$

Hence, $\operatorname{Im} L_v^{(1+m)} = \operatorname{Im} L_{v+m}^{(1)}$.
For the cases $1 \leqslant k < v$,

(1) if $1 \leqslant m \leqslant v-k$, then since $\operatorname{Ker} L_m^{(1)} = \{0\}$ for $1 \leqslant m < v$, we have $Y_k = \cdots = Y_{k+m-1} = 0$ and then $\operatorname{Im} L_k^{(1+m)} = \operatorname{Im} L_{k+m}^{(1)}$ holds;

(2) if $m > v-k$, then $Y_k = \cdots = Y_{v-1} = 0$, $Y_j = eV_j$, $j = v, v+1, ..., k+m-1$, where e is any real number, and then similarly to the case $k = v$, we also have $\operatorname{Im} L_k^{(1+m)} = \operatorname{Im} L_{k+m}^{(1)}$.

THEOREM 3.11. *If β/α^2 is not an algebraic number, then the unique normal form of Eq.* (7) *can be taken as the following form*:

$$\dot{x} = y,$$
$$\dot{y} = \alpha x^\nu y + \beta x^{2\nu+1} + b_{2\nu} x^{2\nu} y + \sum_{m=2\nu+2}^{\infty} a_m x^m + \sum_{\substack{n=\nu+1,\\ n(\mathrm{mod}(\nu+1)) \neq \nu-1,\, \nu}}^{\infty} b_n x^n y, \tag{12}$$

where a_m, b_n are all uniquely determined by Eq. (7).

Remark 3.12. For the case $\nu = 1$, [KOW] proves the uniqueness of the first order normal form. But a term of degree $2\nu = 2$ is missing in the final unique normal form. In fact $\mathrm{Ker}\, L_k^{(1)} = \{0\}$ doesn't hold for $k = 1$.

Remark 3.13. For the case $\mu = 2\nu$, [BS2] has done some basic calculation and gives some incomplete results.

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